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Fine structure of the zeros of orthogonal polynomials, II. OPUC with competing exponential decay

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Abstract

We present a complete theory of the asymptotics of the zeros of OPUC with Verblunsky coefficients $\alpha_n = \sum_{\ell=1}^{L} C_\ell b_\ell^n + O((b\Delta)^n)$ where $\Delta < 1$ and $|b_\ell| = b < 1$.

Keywords: Zeros; OPUC

1. Introduction

This paper is the second in a series [7,8] that discusses detailed asymptotics of the zeros of orthogonal polynomials with special emphasis on distances between nearby zeros. We will focus here on OPUC, orthogonal polynomials on the unit circle; see [9,2,5,6] for background. The polynomials are described by the recursion coefficients, $\{\alpha_n\}_{n=0}^{\infty}$, called Verblunsky coefficients, that give the monic OPUC, $\Phi_n(z)$, by

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z), \tag{1.1}$$

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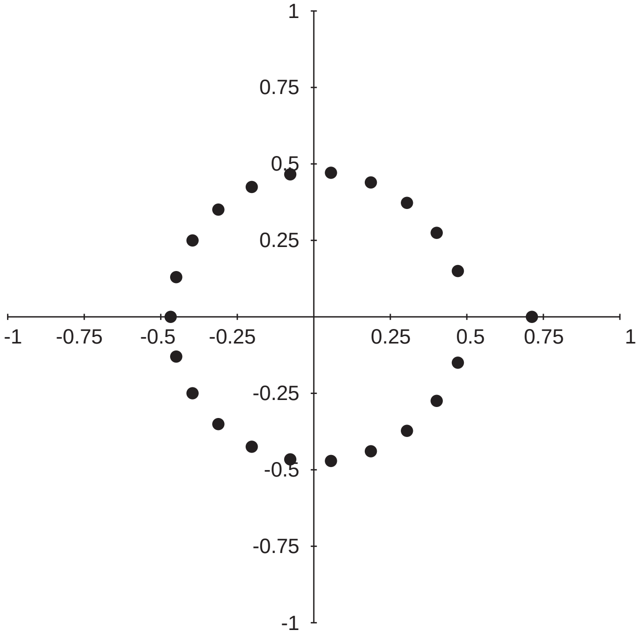

Fig. 1. Zeros when $\alpha_n = \left(\frac{1}{2}\right)^{n+1}$ for $N = 22$.

where

$$\Phi_n^*(z) = z^n\, \overline{\Phi_n(1/\bar{z})}. \tag{1.2}$$

One of the examples discussed in the first paper [7] is where $0 < b < 1$ and

$$\alpha_n = Cb^n + O((b\Delta)^n) \tag{1.3}$$

for $0 < \Delta < 1$. A typical example is shown in Fig. 1 where

$$\alpha_n = (\tfrac{1}{2})^{n+1} \tag{1.4}$$

and zeros of Φ_{22} are shown. Figures and numeric zeros, which appear in [5,7] and here, are computed using Mathematica and code written by M. Stoiciu.

Critical aspects of the zeros in this case are:

(a) Finitely many zeros outside $|z| = b + O(\log n/n)$ at the Nevai–Totik points, that is, solutions of $D(1/\bar{z})^{-1} = 0$ where D is the Szegő function; this is due to Nevai–Totik [4].
(b) No zeros in $|z| < b - O(\log n/n)$. This is due to Barrios–López–Saff [1].
(c) Zeros near $|z| = b$ are asymptotically a distance $2\pi b/n + o(1/n)$ from each other except for a single gap at $z = b$, where the nearby zeros are $be^{\pm 2\pi i/n} + O(1/n^2)$, and the distance between these neighboring zeros is $2(2\pi b/n) + O(1/n^2)$. This is proven in [7].

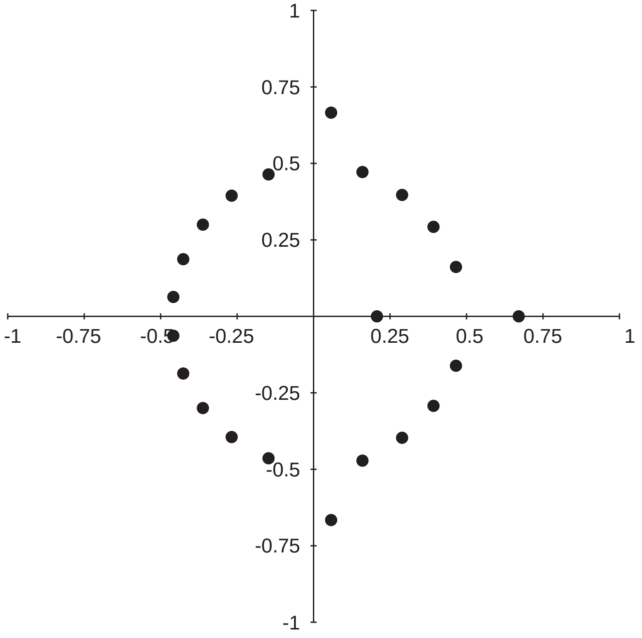

Fig. 2. Zeros when $\alpha_n = \left(\frac{1}{2}\right)^{n+1} \left(1 + 2\cos\left(\frac{\pi}{2}(n+1)\right)\right)$ for $N = 22$.

In this paper, I will analyze the case of Verblunsky coefficients of the form

$$\alpha_n = \sum_{\ell=1}^{L} C_\ell b_\ell^n + O((b\Delta)^n), \tag{1.5}$$

where the b_ℓ's are distinct, $C_\ell \neq 0$ for all ℓ, and

$$|b_\ell| = b \qquad \ell = 1, \ldots, L. \tag{1.6}$$

If the b_ℓ's obey $b_\ell^p = b^p$ for some p, then α_{n+1}/α_n is periodic of period p, and this overlaps examples of BLS [1] discussed later.

A typical example is shown in Fig. 2 where

$$\alpha_n = (\tfrac{1}{2})^{n+1}(1 + 2\cos(\tfrac{\pi}{2}(n+1))) \tag{1.7}$$

and again, zeros of Φ_{22} are shown. This has $b_1 = 1/2$, $b_2 = i/2$, $b_3 = -i/2$.

Here is what happens to (a)–(c) above:

(a′) The Nevai–Totik theory still applies. There are three NT zeros in the example in Fig. 2.
(b′) There are at most $L - 1$ zeros in $|z| < b - O(\log n/n)$ and they can be described explicitly. In the case of periodic α_{n+1}/α_n, the interior zeros for Φ_n with $n = mp+q$ and $m \to \infty$ have explicitly computable limits (limits, not merely accumulation points). For the example in Fig. 2, Φ_n, $n \equiv 2 \pmod 4$, have a zero approaching $\frac{1}{2}(\sqrt{2} - 1) = 0.20710678118\ldots$. The actual zero in Fig. 2 is at $0.20710678374\ldots$.
(c′) There are gaps at each $\bar{b}_\ell$.

The method used in [7] to prove (a)–(c) exploits a second-order difference equation that Φ_n obeys, relating Φ_{n+1} to Φ_n and Φ_{n-1}. That method may extend to the periodic case, $b_\ell^p = b$, but will have small divisor problems if α_n/b^n is only almost periodic. Instead, this paper will use a different and potentially more powerful and illuminating method that views (1.1) with $\Phi_n^*(z)$ fixed as an inhomogeneous first-order difference equation for $\Phi_n(z)$.

In Section 2, we discuss asymptotics of Φ_n for $|z| < b$. In Section 3, we discuss asymptotics of Φ_n in the critical region $b\Delta_1 < |z| < b\Delta_1^{-1}$. In Section 4, we study zeros in $|z| < b$, and in Section 5, the zeros near $|z| = b$. Given Section 2, Section 4 is straightforward. Section 5 will use the ideas in [7]. Finally, Section 6 makes various remarks about the connection to [1].

Figs. 1 and 2 suggest that there might be a connection between the gaps in the clock and the Nevai–Totik zeros since in these two cases the number of NT zeros equals the number of gaps, and the zeros are near the gaps. There is certainly something to this notion. If a coefficient C_ℓ in (1.5) is changed from zero to nonzero, for large n, the zero that was in the gap at zero value of C_ℓ must stop being on the critical circle, and so presumably turns into an NT zero (this is an expectation, not a proof, since a proof involves controlling an interchange of limits). But the connection is not always there. Fig. 8.3 in [5] shows an example where there is a gap but no associated NT zero. Moreover, the discussion in Section 13 of [7] makes it clear that the number of NT zeros can be arbitrary and is not, in general, the same as the number of gaps in the clock.

2. Asymptotics in $|z| < b - \varepsilon$

Our main goal here is to give asymptotics of $\varphi_n(z)$ in the region $|z| < b$ in case (1.5) holds. Our methods will also allow us to say something when weaker asymptotics (ratio asymptotics) holds and, in particular, to improve a result of BLS [1].

We begin with an analysis of some bounds and rate of convergence of Φ_n^*. Estimates similar to these appear in [4,3,1,7]. We use q rather than b since sometimes $q = b + \varepsilon$.

Proposition 2.1. *Suppose that*

$$|\alpha_n| \leqslant Cq^n \tag{2.1}$$

for some $q \in (0, 1)$. *Then*

(i) *With* $C_1 = \prod_{j=0}^\infty (1 + Cq^j) < \infty$, *we have*

$$|z| \leqslant 1 \Rightarrow |\Phi_n^*(z)| \leqslant C_1, \tag{2.2}$$

$$|z| \geqslant 1 \Rightarrow |\Phi_n(z)| \leqslant C_1|z|^n. \tag{2.3}$$

(ii) *For* $1 \leqslant |z| < q^{-1}$,

$$|\Phi_n^*(z)| \leqslant 1 + C_1C|z|(1 - q|z|)^{-1}. \tag{2.4}$$

(iii) *For any* $q' > q$, *there is* $C_{q'}$ *with*

$$|\Phi_n(z)| \leqslant C_{q'}(\max(|z|, q'))^n \tag{2.5}$$

for $|z| < 1$.

(iv) *For* $|z| \leqslant q'$,

$$|\Phi_n^*(z) - D(z)^{-1}D(0)| \leqslant \tilde{C}_{q'}(qq')^n. \tag{2.6}$$

(v) $D(z)^{-1}$ *has an analytic continuation to* $\{z \mid |z| < q^{-1}\}$, *and in that region*,

$$\Phi_n^*(z) \to D(0)D(z)^{-1}. \tag{2.7}$$

(vi) *For* $|z| > q$,

$$\lim_{n\to\infty} z^{-n}\Phi_n(z) = D(0)\,\overline{D(1/\bar{z})}^{-1}. \tag{2.8}$$

Proof. (i) For $|z| = 1$, $|\Phi_n(z)| = |\Phi_n^*(z)|$, so (2.2) holds by induction from (1.1). (2.2) for $|z| < 1$ follows from the maximum principle. (2.3) then follows from (1.2).

(ii) By (2.3) and the * of (1.1),

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z), \tag{2.9}$$

$$|\Phi_{n+1}^*(z)| \leqslant 1 + C_1 \sum_{j=0}^{n} |\alpha_j|\,|z|^{n+1}$$
$$\leqslant 1 + CC_1|z| \sum_{j=0}^{\infty} q^n|z|^n$$

proving (2.4).

(iii) By (1.2) and (2.4), we have (2.5) for $q' \leqslant |z| \leqslant 1$. The result for $|z| < q'$ then follows from the maximum principle.

(iv) By (2.9) and (2.5),

$$\sum_{m=n}^{\infty} |\Phi_{m+1}^*(z) - \Phi_m^*(z)| \leqslant \sum_{m=n}^{\infty} C_q C q^m |z|(q')^m$$
$$\leqslant \tilde{C}_{q'}(qq')^n.$$

Since $\Phi_m^*(z) \to D(0)D(z)^{-1}$, (2.6) holds.

(v) Following [4], we note that (2.9) implies that if $|z| < q^{-1}$, then $\sum_n |\Phi_{n+1}^*(z) - \Phi_n^*(z)| < \infty$, so $\Phi_n^*(z)$ has a limit. Since Szegő's theorem holds if $|z| < 1$, we conclude that $D(z)^{-1}$ has a continuation and (2.7) holds.

(vi) is immediate from (v) and (1.2). □

By iterating (1.1), we obtain

$$\Phi_n(z) = z^n - \sum_{j=1}^{n} \bar{\alpha}_{n-j} z^{j-1}\Phi_{n-j}^*(z). \tag{2.10}$$

We conclude that

Theorem 2.2. *Suppose* (1.5) *holds. Then, uniformly in each disk* $|z| < b - \varepsilon$, *we have that*

$$\varphi_n(z) - \left[\sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1}\right] D(z)^{-1}$$
$$= O(n(b\Delta)^n) + O(nb^{2n}) + O\left(nb^n\left(1 - \frac{\varepsilon}{b}\right)^n\right). \tag{2.11}$$

In particular, uniformly on the disk,

$$\lim_{n\to\infty} |b^{-n}\varphi_n - \tilde{Q}_n(z)| = 0,$$

where

$$\tilde{Q}_n(z) = \left[\sum_{\ell=1}^{L} \bar{C}_\ell \omega_\ell^n (\bar{b}_\ell - z)^{-1}\right] D(z)^{-1} \tag{2.12}$$

and

$$\omega_\ell = \frac{\bar{b}_\ell}{b}. \tag{2.13}$$

Remarks. 1. In (2.11), the b^{2n} is actually $b^{(3-\delta)n}$ for any $\delta > 0$.

2. $D(z)\tilde{Q}_n(z) \prod_{\ell=1}^{L} (\bar{b}_\ell - z)$ is a polynomial of degree $L - 1$ with almost periodic coefficients (periodic if $b_\ell^p = 1$ for some p and all ℓ).

Proof. We begin by noting that

$$\sum_{j=1}^{\infty} \bar{b}_\ell^{n-j} z^{j-1} = \bar{b}_\ell^{n-1}(1 - z\bar{b}_\ell^{-1})^{-1}$$
$$= \bar{b}_\ell^n (\bar{b}_\ell - z)^{-1} \tag{2.14}$$

and by an identical calculation,

$$\sum_{j=n+1}^{\infty} \bar{b}_\ell^{n-j} z^{j-1} = z^n (\bar{b}_\ell - z)^{-1}. \tag{2.15}$$

It follows by (2.10) that if $|z| < b - \varepsilon$, then

$$\left|\Phi_n(z) - D(z)^{-1} D(0) \sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1}\right|$$
$$\leqslant C_\varepsilon |z|^n + \sum_{j=1}^{n} |z|^{j-1} \left|\bar{\alpha}_{n-j} \Phi_{n-j}^*(z) - \sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^{n-j} D(0) D(z)^{-1}\right|. \tag{2.16}$$

The $|z|^n$ term is $O(b^n(1-\frac{\varepsilon}{b})^n)$. By (1.5) and (2.5), the term in $|\cdot|$ in (2.16) is bounded by

$$C[b^{2(n-j)} + (b\Delta)^{n-j}], \tag{2.17}$$

where the first term comes from $|[\Phi_n^* - D(0)D(z)^{-1}]\alpha_n|$ and the second from $|D(z)^{-1}D(0)(\alpha_n - \sum_{\ell=1}^{L} C_\ell b_\ell^n)|$. Since $|z| < b$, the sum in (2.16) is thus bounded by $nC'[b^{2n} + \max(b\Delta, z)^n]$. It follows that

$$D(0)^{-1}\Phi_n(z) - \left[\sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z-\bar{b}_\ell)^{-1}\right] D(z)^{-1} = \text{RHS (right-hand side) of (2.11).}$$

Eq. (2.11) then follows from

$$\begin{aligned} D(0)^{-1}\Phi_n(z) &= \varphi_n(z) + O\left(b^n \left|\prod_{j=1}^{n-1}(1-|\alpha_j|^2)^{1/2} - D(0)\right|\right) \\ &= \varphi_n(z) + O(b^{2n}). \qquad \square \end{aligned}$$

From (2.10), we also get a result that only depends on ratio asymptotics:

Theorem 2.3. *Suppose that α_n is a sequence of Verblunsky coefficients and n_j a subsequence so that*

(i)

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = b \in (0,1), \tag{2.18}$$

(ii)

$$\liminf_{j\to\infty} |\alpha_{n_j}|^{1/n_j} = b. \tag{2.19}$$

(iii) *For all $k = 0, 1, 2, \ldots$ and suitable $\beta_k \in \mathbb{C}$,*

$$\lim_{j\to\infty} \frac{\bar{\alpha}_{n_j-k-1}}{\bar{\alpha}_{n_j-1}} = \beta_k \tag{2.20}$$

exists.

(iv) *For every ε, there is C_ε so that for all j and $k = 1, 2, \ldots$, we have*

$$|\alpha_{n_j-k-1}|(b-\varepsilon)^k \leqslant C_\varepsilon |\alpha_{n_j-1}|. \tag{2.21}$$

Then for $|z| < b$, $\sum_{j=0}^{\infty} \beta_j z^j$ converges absolutely and uniformly on each disk $|z| < b-\varepsilon$ and

$$\lim_{j\to\infty} \frac{\Phi_{n_j}(z)}{\bar{\alpha}_{n_j-1}} = -D(z)^{-1}\sum_{k=0}^{\infty} \beta_k z^k. \tag{2.22}$$

Proof. By (2.21), we have

$$|\beta_k| \leqslant C_\varepsilon (b-\varepsilon)^{-k} \tag{2.23}$$

proving that $\sum_{j=0}^{\infty} \beta_j z^j$ has radius of convergence at least b.

In (2.10) divided by α_{n_j-1}, the summand is bounded by

$$|z|^{j-1}(b-\varepsilon)^{-j} \sup_{m,\,|z|\leqslant b} |\Phi_m^*(z)|$$

which is summable for $|z| < b-\varepsilon$, so by the dominated convergence theorem, (2.22) holds. □

Corollary 2.4. *Let α_n be a sequence of Verblunsky coefficients so that*

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = b \in (0,1). \tag{2.24}$$

Then uniformly for $|z| < b-\varepsilon$,

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{\bar{\alpha}_{n-1}} = -D(z)^{-1}\left(1-\frac{z}{b}\right)^{-1}. \tag{2.25}$$

Remarks. 1. By rotational covariance, if (2.24) holds for some $b \in \mathbb{D}$, we can find a rotated problem with the ratio in (0, 1), so this implies a result whenever ratio asymptotics holds.

2. This is related to results of BLS [1]; see the discussion in Section 6.

Proof. Clearly, (2.20) holds with $\beta_k = b^{-k}$ so we need only prove (2.21). For any δ, we have

$$\left|\frac{\alpha_{m-1}b}{\alpha_m}\right| \leqslant C_m^{(\delta)}(1+\delta),$$

where $C_m^{(\delta)} = 1$ for $m \geqslant M_\delta$ for some M_δ. It follows that

$$\left|\frac{\alpha_{m-k}b^k}{\alpha_m}\right| \leqslant \left[\prod_{m=1}^{M_\delta} C_m^{(\delta)}\right](1+\delta)^k$$

which implies (2.21). □

Similarly, we obtain

Corollary 2.5. *Let α_n be a sequence of Verblunsky coefficients, $b \in (0,1)$, and $c_1, c_2, \ldots, c_p$ a sequence so that*

$$\prod_{j=1}^{p} c_j = 1 \tag{2.26}$$

and

$$\lim_{m\to\infty} \frac{\alpha_{mp+\ell}}{\alpha_{mp+\ell-1}} = bc_\ell \qquad \ell = 1, 2, \ldots, p. \tag{2.27}$$

Then uniformly for $|z| < b - \varepsilon$,

$$\lim_{m\to\infty} \frac{\Phi_{mp+\ell}(z)}{\bar{\alpha}_{mp+\ell-1}} = -D(z)^{-1}G_\ell(z),$$

where

$$G_\ell(z)\left(1 - \frac{z^\ell}{b^\ell}\right) = 1 + (bc_{\ell-1})^{-1}z + (bc_{\ell-1})^{-1}(bc_{\ell-2})^{-1}z^2 + \cdots + \prod_{j=1}^{p-1}(bc_{\ell-j})^{-1}z^{p-1}. \tag{2.28}$$

One can also say something when $b = 1$ if we also have

$$\lim_{n\to\infty} \alpha_n = 0. \tag{2.29}$$

A key issue is that it may not be true that $\sum_{n=1}^\infty |\alpha_n|^2 < \infty$, so $D(z)$ may not exist.

Theorem 2.6. *Let* α_n *be a sequence of Verblunsky coefficients and* $c_1, c_2, \ldots, c_p$ *is a sequence so that* (2.26) *holds and*

$$\lim_{m\to\infty} \frac{\alpha_{mp+\ell}}{\alpha_{mp+\ell-1}} = c_\ell \qquad \ell = 1, 2, \ldots, p. \tag{2.30}$$

Then, uniformly in $|z| < 1 - \varepsilon$,

$$\lim_{m\to\infty} \frac{\varphi_{mp+\ell}(z)}{\bar{\alpha}_{mp+\ell-1}\varphi^*_{mp+\ell}(z)} = -G_\ell(z), \tag{2.31}$$

where G is given by (2.28) *with* $b = 1$.

Proof. Eqs. (2.29) and (2.9), together with $|\Phi_n(z)| \leqslant |\Phi_n^*(z)|$ on $\bar{\mathbb{D}}$, implies that on $\bar{\mathbb{D}}$,

$$\lim_{n\to\infty} \frac{\Phi_n^*(z)}{\Phi_{n+1}^*(z)} = 1.$$

Dividing (2.10) by $\bar{\alpha}_{mp+\ell-1}\Phi^*_{mp+\ell}$, we obtain the result by the same argument that led to Corollary 2.5. □

3. Asymptotics in the critical region

In this section, we will determine asymptotics of $\Phi_n(z)$ in an annulus about $|z| = b$ when (1.5) holds. The idea will be to view (1.1) as an inhomogeneous equation, so we first look at some solutions with particular inhomogeneities. Define for $z \neq \bar{b}_\ell$ and $n = 0, 1, 2, \ldots,$

$$u_n^{(\ell)} = \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1}. \tag{3.1}$$

Proposition 3.1. $u_n^{(\ell)}$ *obeys*

$$u_{n+1}^{(\ell)} = z u_n^{(\ell)} - \bar{b}_\ell^n \tag{3.2}$$

for all $z \in \mathbb{C}$, $z \neq \bar{b}_\ell$, *and all* $n = 0, 1, 2, \ldots.$

Proof. $u_{n+1}^{(\ell)} - z u_n^{(\ell)} = (\bar{b}_\ell^n - z) u_n^{(\ell)} = -\bar{b}_\ell^n$. $\square$

Next, define

$$R_n(z) = \bar{\alpha}_n \Phi_n^*(z) - \sum_{\ell=1}^{L} \bar{C}_\ell b_\ell^n D(z)^{-1} D(0) \tag{3.3}$$

and also define

$$s_n(z) = \sum_{j=0}^{\infty} z^{-j-1} R_{n+j}(z) \tag{3.4}$$

We have

Proposition 3.2. *Let* α_n *obey* (1.5). *Then there is* $\Delta_1 < 1$,

(i)

$$\sup_{|z| \leqslant 1} |R_n(z)| \leqslant C (b\Delta_1)^n. \tag{3.5}$$

(ii) *The sum in* (3.4) *converges uniformly in*

$$\mathbb{A} = \{z \mid 1 > |z| > b\Delta_1\} \tag{3.6}$$

and $s_n(z)$ *is analytic there.*

(iii) *We have in* $\mathbb{A}$ *that*

$$|s_n(z)| \leqslant C (b\Delta_1)^n (|z| - b\Delta_1)^{-1}. \tag{3.7}$$

(iv) s_n *obeys*

$$s_{n+1}(z) = z s_n(z) - R_n(z). \tag{3.8}$$

Proof. (i) follows from (1.5) and (2.6).

(ii), (iii) Since

$$|z^{j-1} R_{n+j}(z)| \leqslant |z|^{-1} (b\Delta_1)^n (|z|^{-1} b\Delta_1)^j$$

we have a geometric series which yields (ii) and (iii).

(iv) Since the sum converges absolutely,

$$\begin{aligned} s_{n+1}(z) - zs_n(z) &= \sum_{j=1}^{\infty} z^{-j} R_{n+j}(z) - \sum_{j=0}^{\infty} z^{-j} R_{n+j} \\ &= -R_n(z). \qquad \square \end{aligned}$$

The main result of this section is

Theorem 3.3. *Let* α_n *obey* (1.5). *Then for some* $\Delta_1 < 1$ *and* $z \in \mathbb{A}$ *given by* (3.7), *we have that*

$$\Phi_n(z) = s_n(z) + \left[\sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(0) D(z)^{-1} + z^n D(0) \, \overline{D(1/\bar{z})}^{-1}. \qquad (3.9)$$

Remarks. 1. Since $\varphi_n = \kappa_n \Phi_n(z)$ and $\kappa_n = D(0)^{-1}(1 + O(b^n))$, this also gives us asymptotics for φ_n.

2. Since Φ_n is analytic in $\mathbb{A}$, the poles at $\bar{b}_\ell$ in the second and third terms of (3.9) must cancel.

3. In $\mathbb{A}$, (3.7) implies s_n is small compared to both z^n and b^n, so the asymptotics of Φ_n comes from the competition between the second and third terms in (3.9).

Proof. Let

$$Q_n(z) = \Phi_n(z) - s_n(z) - \left[\sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(0) D(z)^{-1}.$$

By (1.1), (2.2), and (3.8), we have

$$Q_{n+1}(z) = zQ_n(z)$$

so

$$Q_n(z) = f(z) z^n.$$

Since Q_n is analytic in $\mathbb{A} \setminus \{\bar{b}_\ell\}_{\ell=0}^{L}$, $f(z)$ is analytic there.

By (3.7),

$$\lim_{n\to\infty} |z|^{-n} |s_n(z)| = 0$$

in $\mathbb{A}$, and if $|z| > b$, $|z|^{-n} \sum_{\ell=1}^{L} \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \to 0$, so for $|z| > b$,

$$\begin{aligned} f(z) &= \lim_{n\to\infty} z^{-n} Q_n(z) = \lim_{n\to\infty} z^{-n} \Phi_n(z) \\ &= D(0) \, \overline{D(1/\bar{z})}^{-1} \end{aligned}$$

by (2.8). $\square$

4. Zeros in $|z| < b - \varepsilon$

In this section, we use the asymptotic result from Section 2 to analyze zeros of φ_n in the region where $|z| < b$. We initially focus on the case where (1.5) holds. A key role is played by the polynomials

$$P_n(z) = \sum_{\ell=1}^{L} \bar{C}_\ell \omega_\ell^n \prod_{k \neq \ell} (z - \bar{b}_k) \tag{4.1}$$

of degree at most $L - 1$. Here $\omega_\ell = \bar{b}_\ell / b$.

The P_n are almost periodic in n and, in particular, for any sequence n_j, there is a subsequence $n_{j(k)}$ so $P_\infty \equiv \lim\ P_{n_{j(k)}}$ exists and is a nonzero polynomial (since $P_\infty / \prod_\ell (\bar{b}_\ell - z)$ has poles at each $\bar{b}_\ell$).

Theorem 4.1. *Let* (1.5) *hold. Then for any* $\varepsilon > 0$, *there is an* N *so that for* $n \geqslant N$, $\varphi_n(z)$ *has at most* $L - 1$ *zeros in* $\{z \mid |z| < b - \varepsilon\} \equiv \mathbb{S}$.

Proof. If not, we can find a sequence $n(j) \to \infty$ so that $P_{n(j)}(z)$ has at least L zeros in $\bar{\mathbb{S}}$. By passing to a further subsequence, we can suppose $P_{n(j)} \to P_\infty$ and that the L zeros have limits $z_1, \ldots, z_L$ in $\bar{\mathbb{S}}$ (maybe not distinct). By Theorem 2.2,

$$\lim_{j \to \infty} \varphi_{n(j)} D(z) \prod_{\ell=1}^{L} (z - \bar{b}_\ell) = P_\infty(z) \tag{4.2}$$

in a neighborhood of $\bar{\mathbb{S}}$, so by Hurwitz's theorem, P_∞ has L zeros (counting multiplicity). Since P_∞ has degree $L - 1$ and is not identically zero, we have a contradiction. □

Using Hurwitz's theorem and (4.2), we also have an existence result for zeros:

Theorem 4.2. *Let* (1.5) *hold and let* $\omega_\ell = \bar{b}_\ell / b$. *Suppose* $n(j)$ *is a subsequence so that* $\lim\ \omega_\ell^{n(j)}$ *exists, call it* $\omega_\ell^{(\infty)}$. *Let*

$$P_\infty(z) = \sum_{\ell=1}^{L} \bar{C}_\ell \omega_\ell^{(\infty)} \prod_{k \neq \ell} (z - \bar{b}_k) \tag{4.3}$$

and let $\{w_j\}_{j=1}^{J}$ *be its zeros in* $\{z \mid |z| < b\}$. *Then for all sufficiently small* δ *and* $j \geqslant N_\delta$, $\varphi_{n(j)}(z)$ *has one zero within* δ *of each* w_j *and no other zero in* $\{z \mid |z| < b - \delta\}$.

Remark. By "one zero within δ of w_j," we actually mean exactly k zeros if some w_j occurs k times in the list of zeros counting multiplicity.

Since the right side of (2.25) is nonvanishing on $\{z \mid |z| < b\}$, we recover a result of BLS [1] from Corollary 2.4, Theorem 2.6, and Hurwitz's theorem:

Theorem 4.3. *Let α_n be a sequence of Verblunsky coefficients so that*

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = b \in (0, 1].$$

Then for any $\varepsilon > 0$, there is an N_ε so $\varphi_n(z)$ has no zeros in $\{z \mid |z| < b - \varepsilon\}$ if $n \geqslant N_\varepsilon$.

Finally, Corollary 2.5 and Hurwitz's theorem imply

Theorem 4.4. *Let α_n be a sequence of Verblunsky coefficients, $b \in (0, 1)$, and $c_1, c_2, \dots, c_p$ a sequence so that* (2.26) *and* (2.27) *hold. Let G_ℓ be given by* (2.28), *let $W_\ell = G_\ell(1 - z^\ell/b^\ell) =$ RHS of* (2.28), *and let $\{w_j^{(\ell)}\}_{j=1}^{N_\ell}$ be the zeros of W_ℓ in $\{z \mid |z| < b\}$. Then for any sufficiently small δ, there is an N so for $mp + \ell \geqslant N$, we have that the only zeros of $\varphi_{mp+\ell}$ in $\{z \mid |z| < b - \delta\}$ are one each within δ of each $w_j^{(\ell)}$.*

Remark. As we will explain in Section 6, that the only possible limit points of zeros are the $w_j^{(\ell)}$ is a result of BLS [1], but they do not prove there actually are zeros there.

Example 4.5. Let α_n be given by (1.7). We have $b = \frac{1}{2}$, $p = 4$, and $c_1 = -1$, $c_2 = -1$, $c_3 = 3$, $c_4 = \frac{1}{3}$. Thus

$$W_2(z) = 1 - 2z - 12z^2 - 8z^3,$$

which has zeros at $-\frac{1}{2}$ and at $\frac{1}{2}(-1 \pm \sqrt{2})$. Only $(\sqrt{2} - 1)/2$ is in $\{z \mid |z| < \frac{1}{2}\}$. The comparison of the limit and the zeros of Φ_{22} appears in Section 1 just after Fig. 2. It is not coincidental that W_2 has a zero at $z = -\frac{1}{2}$. In this case, the second term in (3.9) is, for $n \equiv 2 \pmod 4$, $C(\frac{1}{2})^n W_2(z)/(z^4 - \frac{1}{16})$ with poles only at $\frac{1}{2}, \pm\frac{1}{2}i$. The potential pole at $z = -\frac{1}{2}$ has to be cancelled by a zero in W_2. $\square$

As in [7], one can analyze how close the zeros of φ_n are to the points $w_j^{(\ell)}$. In general, they are exponentially close. If the $w_j^{(\ell)}$ are in the annulus where (3.9) holds, one can write down the leading asymptotic exactly. For example, if $w_j^{(\ell)}$ is a k-fold zero and $D(1/\bar{w}_j(z)) \neq 0$, then the zeros have a clock structure as in Theorem 4.5 of [7].

By using Theorem 2.6, we see that Theorem 4.4 extends to the case $b = 1$ if $\lim\, \alpha_n = 0$. In particular, if $\lim\, \alpha_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ (e.g., $\alpha_n = (n + 2)^{-\beta}$), then there are no zeros of φ_n in $\{z \mid |z| < 1 - \delta\}$ for n large.

5. Zeros in the critical region

Given Theorem 3.3 and the estimate (3.7), the analysis of zeros of φ_n in the region $\{z \mid b\Delta_1 < |z| < b\Delta_1^{-1}\}$ is identical to the analysis in [7] of the zeros in case $L = 1$. The gap in that case if $z = b$ comes from the analysis of $\overline{D(1/\bar{z})\ \varphi_n(z)}$ which has zeros with no gap. The gap in zeros of $\varphi_n(z)$ comes from the fact that $\overline{D(1/\bar{z})}$ has zeros at $z = b$. In our case, when (1.5) holds, $\overline{D(1/\bar{z})}$ has a zero at each $\bar{b}_\ell$, so there are gaps at all those points.

The following extends Theorem 4.3 of [7] and has the same proof:

Theorem 5.1. *Let α_n be a sequence of Verblunsky coefficients obeying* (1.5). *Then for some δ, all the zeros $\{z_j^{(n)}\}_{j=1}^{N_n}$ of $\varphi_n(z)$ with $||z| - b| < \delta$ obey*

(1)

$$\sup_j ||z_j^{(n)}| - b| = O\left(\frac{\log n}{n}\right). \tag{5.1}$$

(2) *For n large, the $z_j^{(n)}$ can be ordered in increasing arguments and*

$$\frac{|z_{j+1}^{(n)}|}{|z_j^{(n)}|} = 1 + O\left(\frac{1}{n \log n}\right). \tag{5.2}$$

(3) *Let $\{\tilde{z}_j^{(n)}\}_{j=1}^{N_n+L}$ be the sequence of $z_j^{(n)}$'s with L points added at $\{\bar{b}_\ell\}_{\ell=1}^{L}$ still listed in increasing order.*

Then

$$\arg z_{j+1}^{(n)} - \arg z_j^{(n)} = \frac{2\pi}{n} + O\left(\frac{1}{n \log n}\right) \tag{5.3}$$

for $j = 1, 2, \ldots, N_n + L$ with $\arg z_{N_n+L+1}^{(n)} \equiv 2\pi + \arg z_1^{(n)}$. Moreover, if $D(z)^{-1}$ is nonvanishing on $\{z \mid |z| = b^{-1}\}$, then $O(1/n \log n)$ in (5.2) *and* (5.3) *can be replaced by $O(1/n^2)$, and $O(\log n/n)$ in* (5.1) *can be replaced by $O(1/n)$.*

Remark. In particular, the zeros nearest $\bar{b}_\ell$ are $\bar{b}_\ell e^{\pm 2\pi i/n} + O(1/n^2)$ with the difference in the args equal to $4\pi/n + O(1/n^2)$.

6. Connection to the results of Barrios–López–Saff

In this final section, we want to relate the results of [1] to ours. In their work, determinants of the following form enter:

$$\Delta_m(z) = \begin{vmatrix} z + x_1 & zx_2 & 0 & & \\ 1 & z + x_2 & zx_3 & \ddots & \\ 0 & 1 & z + x_3 & \ddots & \ddots \\ & & \ddots & \ddots & zx_m \\ & & & 1 & z + x_m \end{vmatrix}, \tag{6.1}$$

where we also define $\Delta_0(z) \equiv 1$. We need the following:

Proposition 6.1. (i) *For $m = 2, 3, \ldots$*

$$\Delta_m(z) = (z + x_m)\Delta_{m-1} - zx_m\Delta_{m-2}(z). \tag{6.2}$$

(ii) *For* $m = 1, 2, \ldots$

$$\Delta_m(z) = z\Delta_{m-1} + x_1 x_2 \ldots x_m. \tag{6.3}$$

(iii) *For* $m = 1, 2, \ldots$

$$\Delta_m(z) = z^m + x_1 z^{m-1} + x_1 x_2 z^{m-2} + \cdots + x_1 \ldots x_m. \tag{6.4}$$

Proof. (i) Eq. (6.2) comes from expanding Δ_m in minors along the last row.

(ii) Eq. (6.2) reads

$$\Delta_m(z) = z\Delta_{m-1}(z) + x_m[\Delta_{m-1}(z) - z\Delta_{m-2}],$$

which implies (6.3) inductively once one notes that (6.3) holds for $m = 1$ since $\Delta_1(z) = z + x_1$.

(iii) This follows by induction from (6.3). □

In [1], they consider sequences of Verblunsky coefficients where (2.26) and (2.27) hold to prove that the only accumulation points of zeros of $\varphi_{mp+\ell}$ are given by zeros of a polynomial that has the form of (6.1). Using (6.4) and

$$(x_1 \ldots x_m)^{-1}\Delta_m(z) = 1 + x_m^{-1}z + x_m^{-1}x_{m-1}^{-1}z^2 + \cdots + (x_1 \ldots x_m)^{-1}z^m$$

one sees their polynomials are up to a constant, our polynomial W_ℓ. Thus our results extend theirs (in that we prove there are, in fact, always limit points).

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